COMPUTING MULTI-POINT SESHADRI CONSTANTS ON $\mathbf{P}^2$

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ABSTRACT. We describe an approach for computing arbitrarily accurate estimates for multi-point Seshadri constants for n generic points of $\mathbf{P}^2$. We apply the approach to obtain improved estimates. We work over an algebraically closed field of characteristic 0.

1. INTRODUCTION

Given a positive integer n, the codimension 1 multipoint Seshadri constant for points $p_1, \ldots, p_n$ of $\mathbf{P}^N$ is the real number

$$\varepsilon(N, p_1, \ldots, p_n) = {}^{N-1}\!\sqrt{\inf\left\{\frac{\deg(Z)}{\Sigma_{i=1}^n \mathrm{mult}_{p_i} Z}\right\}},$$

where the infimum is taken with respect to all hypersurfaces Z, through at least one of the points ([De], [S]). It is well known and not difficult to prove that $\varepsilon(N, p_1, \ldots, p_n) \leq 1/\sqrt[N]{n}$, but lower bounds are much more challenging (see [N], [Ku] and [X2]). We also take $\varepsilon(N, n)$ to be defined as $\sup\{\varepsilon(N, p_1, \ldots, p_n)\}$, where the supremum is taken with respect to all choices of n distinct points p_i of $\mathbf{P}^N$. It is not hard to see that $\varepsilon(N, n) = \varepsilon(N, p_1, \ldots, p_n)$ for *very general* points $p_1, \ldots, p_n$ (i.e., in the intersection of countably many Zariski-open and dense subsets of $(\mathbf{P}^N)^n$); some results suggest that the equality might hold in fact for *general* points, i.e., in a Zariski-open subset of $(\mathbf{P}^N)^n$ (see [O], [S]). In this work we shall describe a an approach for obtaining arbitrarily accurate lower bounds for the Seshadri constant $\varepsilon(2, p_1, \ldots, p_n)$ that hold for general points p_i and thus bound also $\varepsilon(2, n)$. We will hereafter denote $\varepsilon(2, n)$ simply by $\varepsilon(n)$.

The method we will use depends on ruling out the occurrence of so-called abnormal curves. Given generic points $p_1, \ldots, p_n \in \mathbf{P}^2$, let $\pi : X \to \mathbf{P}^2$ denote the birational morphism given by blowing up the points. The divisor class group $\mathrm{Cl}(X)$ of X has $\mathbf{Z}$-basis given by the classes $L, E_1, \ldots, E_n$, where L is the pullback of the class of a line and E_i is the class of $\pi^{-1}(p_i)$. The intersection form on $\mathrm{Cl}(X)$ is a bilinear form with respect to which $L, E_1, \ldots, E_n$ is orthogonal with $-L^2 = E_1^2 = \cdots = E_n^2 = -1$. We say that a divisor, or a divisor class, F is nef if $F \cdot C \geq 0$ for every effective divisor C. This terminology provides an alternate description of Seshadri constants: $\varepsilon(n)$ is the supremum of all real numbers t such that $F = (1/t)L - (E_1 + \cdots + E_n)$ is nef.

Now let C be an effective divisor on X whose class $[C] = dL - m_1 E_1 - \cdots - m_n E_n$ satisfies $d\sqrt{n} < m_1 + \cdots + m_n$. Nagata [N] calls such a curve C an *abnormal* curve (with respect to the given value of n). Nagata also found all curves abnormal for each $n < 10$, showed no curve is abnormal for n when n is a square and conjectured there are no abnormal curves for $n \geq 10$. If F is the $\mathbf{R}$-divisor class $\sqrt{n}L - E_1 - \cdots - E_n$, then an effective divisor C is abnormal if and only if $F \cdot C < 0$. More generally, given *delta* ≥ 0, let $F(\delta) = d'L - E_1 - \cdots - E_n$ where $d' = \sqrt{(n+\delta)}$; note that $F(\delta)^2 = \delta$. If C is a the class of a reduced irreducible curve such that $F(\delta) \cdot C < 0$, we say that C is $F(\delta)$-abnormal. In particular, an $F(0)$-abnormal class is abnormal.

Our interest in the case of reduced irreducible curves is because it can be shown (see Remark 2.3) that if $\varepsilon(n) < 1/\sqrt{n}$, then $\varepsilon(n) = d/(m_1 + \cdots + m_n)$ where $[C] = dL - m_1 E_1 - \cdots - m_n E_n$ is the class of *any* reduced irreducible abnormal curve C on X.

Suppose one can somehow produce a set S_n of classes such that if there is an abnormal curve C for n, then S_n contains its class $[C] = dL - m_1E_1 - \cdots - m_nE_n$. We will show in Section 3 how to do this in such a way that the set of ratios $d/(m_1 + \cdots + m_n)$ is well-ordered and *small* in a suitable sense. By taking the minimum of $d/(m_1 + \cdots + m_n)$ over all $[C] \in S_n$, one obtains a lower bound for $\varepsilon(n)$. The more elements $[C] \in S$ one can rule out (by showing that $[C]$ is in fact not the class of a reduced irreducible curve), the better this bound becomes. This approach was used by [X1], [S], [ST] and [T], with the latter obtaining the bound $\varepsilon(n) \geq (1/\sqrt{n})\sqrt{1 - 1/(12n+1)}$.

In order to apply this method to obtain arbitrarily accurate estimates of $\varepsilon(n)$, one must have a way of producing such a set S_n, and listing its elements $[C] = dL - m_1E_1 - \cdots - m_nE_n$ in ascending order of $d/(m_1 + \cdots + m_n)$. This is possible, as we show below, using the results of [HR2], where Seshadri constants of arbitrary surfaces are considered. In particular, the main tool here is Theorem 2.2 below, which is a restatement of Theorem 1.2.1 [HR2] for $X = \mathbf{P}^2$. The following result is a simplified version of Theorem 2.2. Here, $m^{[n]}$ denotes the vector $(m, \ldots, m)$ with n entries, and $\alpha(m^{[n]})$ denotes the least t such that t is the degree of a form vanishing at n general points with order at least m at each point. We also note that we use $\alpha_0(m^{[n]})$ to denote the least t such that t is the degree of an *irreducible* form vanishing at n general points with order m at each point.

Theorem 1.1. *Let $n \geq 10$ and $\mu \geq 1$ be integers, and assume that $\alpha(m^{[n]}) \geq m\sqrt{n}$ for all $1 \leq m < \mu$. Then*

$$\varepsilon(n) > \frac{1}{\sqrt{n}}\sqrt{1 - \frac{1}{(n-2)\mu}},$$

with $\varepsilon(n) = \alpha(\mu^{[n]})/(\mu n)$ if $\alpha(\mu^{[n]}) < \mu\sqrt{n}$.

See Section 2 for the proof.

Remark 1.2. As also noted in [HR2], applying this result using results of [CCMO] and [HR1] already gives lower bounds on $\varepsilon(n)$ which for most n are better than what was known previously. For example, since $\alpha(m^{[n]}) \geq m\sqrt{n}$ for $n \geq 10$ and $m < 21$ by [CCMO], Theorem 1.1 implies that

$$\varepsilon(n) > (1/\sqrt{n})\sqrt{1 - 1/(21n - 42)}.$$

Using [Du], which increases the result of [CCMO] from $m \leq 21$ to $m \leq 42$ gives

$$\varepsilon(n) > (1/\sqrt{n})\sqrt{1 - 1/(42n - 84)}.$$

Moreover, we also have $\alpha(m^{[n]}) \geq m\sqrt{n}$ for $n \geq 10$ and $m \leq \lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor - 3)/2$ (see the proof of Corollary 1.2(a) of [HR1]), so taking $\mu = \lceil(n - 5\sqrt{n} + 4)/2\rceil + 1 = \lceil(\sqrt{n} - 1)(\sqrt{n} - 4)/2\rceil + 1 \leq 1 + \lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor - 3)/2$, Theorem 1.1 implies

$$\varepsilon(n) > \frac{1}{\sqrt{n}}\sqrt{1 - \frac{2}{n^2 - 5n\sqrt{n}}}.$$

In Section 3 we describe our algorithm for obtaining arbitrarily accurate estimates for $\varepsilon(n)$, and we demonstrate its use by obtaining estimates for $\varepsilon(n)$ for all n in the range $10 \leq n \leq 99$. In most cases the estimates we obtain are the best currently known.

One can also get formulas that apply for a range of values of n by analyzing the algorithm in order to approximate the outcome of each stage of the algorithm. The disadvantage of doing this is that the formulas obtained this way give estimates for $\varepsilon(n)$ which are not as good as one can get by applying the algorithm for specific values of n. A compensating advantage is the convenience of having a lower bound for $\varepsilon(n)$ given by a formula in terms of n. Thus in Section 2 we give some such formulas, by applying Theorem 2.2 and results of [HR1].

2. GENERAL RESULTS

We begin by stating formulas giving lower bounds for $\varepsilon(n)$.

Theorem 2.1. *Let $n \geq 10$ be a nonsquare integer, let $d = \lfloor\sqrt{n}\rfloor$ and consider $\Delta = n - d^2 > 0$ (note that $\Delta \leq 2d$).*

(a) *If $\Delta = 1$, then $\varepsilon(n) \geq \frac{1}{\sqrt{n}}\sqrt{1 - \frac{1}{(2n-1)^2}}$.*

(b) If $\Delta = 2$, then $\varepsilon(n) \geq \frac{1}{\sqrt{n}} \sqrt{1 - \frac{1}{n(n-1)}}$.

(c) If $\Delta > 2$ is odd, then $\varepsilon(n) \geq \frac{1}{\sqrt{n}} \sqrt{1 - \frac{1}{n(d(d-3)+1)}} \geq \frac{1}{\sqrt{n}} \sqrt{1 - \frac{1}{n(n-5\sqrt{n}+1)}}$.

(d) If $\Delta > 3$ is even, then $\varepsilon(n) \geq \frac{1}{\sqrt{n}} \sqrt{1 - \frac{2}{n(d(d-3)+2)}} \geq \frac{1}{\sqrt{n}} \sqrt{1 - \frac{2}{n(n-5\sqrt{n}+2)}}$.

(e) If Δ is odd and $2d - 1 > \Delta \geq 4\sqrt[4]{n} + 1$, then $\varepsilon(n) \geq \frac{1}{\sqrt{n}} \sqrt{1 - \frac{1}{n^2}}$.

(f) If $\Delta = 2d - 1$, then $\varepsilon(n) \geq \frac{1}{\sqrt{n}} \sqrt{1 - \frac{2}{n(n\sqrt{n}-5n+5\sqrt{n}-1)}}$.

We remark that the bound $\varepsilon(n) \geq (1/\sqrt{n})(\sqrt{1 - 1/f(n)})$ is equivalent to the inequality $\mathcal{R}_n(L) \leq 1/f(n)$, where $\mathcal{R}_n(L)$ is the *n-th remainder* of the divisor class L, introduced by P. Biran in [Bi]. Note that the larger $f(n)$ is, the better is the bound. For $n \geq 10$, our results show that $f(n)$ can be taken to be the maximum of $42(n-2)$ and a function which is at least quadratic in n. Thus we produce an $f(n)$ which is always larger than the best previous general bound, for which $f(n) = 12n + 1$ [T].

For special values of n, [Bi] also gives bounds better than those of [T], and these bounds are quadratic in n. (For example, if $n = (ai)^2 \pm 2i$ for positive integers a and i, then $f(n) = (a^2 i \pm 1)^2$, and, if $n = (ai)^2 + i$ for positive integers a and i with $ai \geq 3$, then $f(n) = (2a^2 i + 1)^2$.) However, except when $n - 1$ is a square, the bounds of Theorem 2.1 are better for n large enough. (To see this when $n \pm 2$ is a square, make a direct comparison; otherwise, look at coefficients of the n^2 term in $f(n)$.)

Additional bounds are given in [H]; they apply to all values of n and are almost always better than any bound for which $f(n)$ is linear in n (more precisely, given any constant a, let $\nu_a(n)$ be the number of integers i from 1 to n for which $f(i)$ from [H] is bigger than ai; then $\lim_{n \to \infty} \nu_a(n)/n = 1$). Nonetheless, although the bounds in [H] are not hard to compute for any given value of n, they are not explicit or simple enough to make them easy to work with. Moreover, computations for specific values of n (see, for example, Table 2) show in almost all cases that the bounds we obtain here are better than those of [H].

To prove Theorems 1.1 and 2.1 we apply the following result, which is just Theorem 1.2.1 of [HR2], restated in the case of $\mathbf{P}^2$:

Theorem 2.2. *Let $n \geq 10$ be an integer, and $\mu \geq 1$ a real number.*

(a) *If $\alpha(m^{[n]}) \geq m\sqrt{n - \frac{1}{\mu}}$ for every integer $1 \leq m < \mu$, then*

$$\varepsilon(n) > (1/\sqrt{n}) \sqrt{1 - 1/((n-2)\mu)}.$$

(b) *If $\alpha_0(m^{[n]}) \geq m\sqrt{n - \frac{1}{\mu}}$ for every integer $1 \leq m < \mu$, and if $\alpha_0((m^{[n-1]}, m+k)) \geq \frac{mn+k}{\sqrt{n}} \sqrt{1 - \frac{1}{n\mu}}$ for every integer $1 \leq m < \mu/(n-1)$ and every integer k with $k^2 < (n/(n-1)) \min(m, m+k)$, then*

$$\varepsilon(n) \geq \frac{1}{\sqrt{n}} \sqrt{1 - \frac{1}{n\mu}}.$$

Note that Theorem 2.2 is an improved but more technical version of Theorem 1.1. This is clear for Theorem 2.2(a). For Theorem 2.2(b), we can see this, for example, by applying the argument in Remark 1.2 to Theorem 2.2 and using the easy fact that for no k and d with $n \geq 10$ is $dL - (E_1 + \cdots + E_n) - kE_1$ the class of an abnormal curve, to obtain

$$\varepsilon(n) \geq \frac{1}{\sqrt{n}} \sqrt{1 - \frac{1}{21n}} \qquad (^{\circ\circ\circ})$$

for $n \geq 12$. As Table 2 shows, this lower bound holds also for $10 \leq n \leq 11$, and hence for all $n \geq 10$, and thus improves one of the bounds we had obtained in Remark 1.2 from Theorem 1.1.

In preparation for proving Theorem 1.1, we also need the following remark.

Remark 2.3. If $\varepsilon(n) < 1/\sqrt{n}$, we justify the well known fact that there is an irreducible curve C whose class $[C] = dL - m_1 E_1 - \cdots - m_n E_n$ satisfies $d\sqrt{n} < m_1 + \cdots + m_n$, and for any such C we have $\varepsilon(n) = d/(m_1 + \cdots + m_n)$. But if $\varepsilon(n) < 1/\sqrt{n}$, then by definition there is a curve D, perhaps not irreducible, whose class $[D] = aL - b_1 E_1 - \cdots - b_n E_n$ satisfies $a\sqrt{n} < b_1 + \cdots + b_n$, and clearly the class $[C] = dL - m_1 E_1 - \cdots - m_n E_n$ of some irreducible component C of D satisfies $d/(m_1 + \cdots + m_n) \leq a/(b_1 + \cdots + b_n) < 1/\sqrt{n}$. Because the points p_i blown up to give E_i are general, if there exists an irreducible component C with multiplicities m_i, then irreducible components occur for every permutation of the multiplicities. Thus we

may as well assume that $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$. Now suppose C' is also an irreducible curve whose class $[C'] = d'L - m'_1 E_1 - \cdots - m'_n E_n$ satisfies $d'\sqrt{n} < m'_1 + \cdots + m'_n$ and $m'_1 \geq m'_2 \geq \cdots \geq m'_n \geq 0$. Then clearly $dd' < d'(m_1 + \cdots + m_n)/\sqrt{n} < (m'_1 + \cdots + m'_n)(m_1 + \cdots + m_n)/n$. But using $m'_1 \geq m_2 \geq \cdots \geq m'_n \geq 0$ and $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$, it is not hard to check that $(m'_1 + \cdots + m'_n)(m_1 + \cdots + m_n)/n = m'm_1 + \cdots + m'm_n \leq m'_1 m_1 + \cdots + m'_n m_n$, where $m' = (m'_1 + \cdots + m'_n)/n$, and hence that $C' \cdot C < 0$. Thus $C = C'$, so the minimum ratio $a/(b_1 + \cdots + b_n)$ occurs for an irreducible curve, and any irreducible curves for which the ratio is less than $1/\sqrt{n}$ give the same ratio.

We can now prove Theorem 1.1:

Proof of Theorem 1.1. The bound $\varepsilon(n) > (1/\sqrt{n})\sqrt{1 - 1/((n-2)\mu)}$ follows immediately from Theorem 2.2. So all that is required is to justify that $\varepsilon(n) = \alpha(\mu^{[n]})/(\mu n)$, if $\alpha(m^{[n]}) < m\sqrt{n}$ for some m, and μ is the least m such that $\alpha(m^{[n]}) < m\sqrt{n}$. Now let D be a curve such that $[D] = \alpha(\mu^{[n]})L - \mu(E_1 + \cdots + E_n)$. Then as in Remark 2.3, D has an irreducible component C which is abnormal for n, hence $D \cdot C < 0$ and $[C]$ is almost uniform by [S]; i.e., $[C] = dL - b(E_1 + \cdots + E_n) - kE_1$ for some d, b and k. But then the curve C_i whose class is $[C_i] = dL - b(E_1 + \cdots + E_n) - kE_i$ is also an irreducible component of D for each $1 \leq i \leq n$. But $[C_1 + \cdots + C_n] = ndL - (nb+k)(E_1 + \cdots + E_n)$ has $nd < (nb+k)\sqrt{n}$ and $nb + k \leq \mu$, and so by hypothesis $\mu = nb + k$ and $nd = \alpha(\mu^{[n]})$. As in Remark 2.3, $\varepsilon(n) = d/(bn+k)$ since C is irreducible and abnormal, but $d/(bn+k) = nd/(n(nb+k)) = \alpha(\mu^{[n]})/(mn)$, as claimed. $\qquad\square$

We close this section by proving Theorem 2.1.

Proof of Theorem 2.1. In applying Theorem 2.2(b), note that it is always true that $\alpha_0(m^{[n]}) \geq \alpha(m^{[n]})$.

(a) This is the result of [Bi], obtained for $n = (ai)^2 + i$, using $a = d$, $i = 1$ and using $f(n) = a^2 i + 1$.

(b) By Corollary 4.1(b) [HR1], we have $\alpha(m^{[n]}) \geq m\sqrt{n}$ for $10 \leq n$ when $\Delta = 2$ and $m \leq d^2 = n - 2$. Using $\mu = n - 1$, the result is now immediate from Theorem 2.2(b), since there is no integer m in the range $1 \leq m < \mu/(n-1)$.

(c) By Corollary 4.1(a) [HR1], we have $\alpha(m^{[n]}) \geq m\sqrt{n}$ for $10 \leq n$ when Δ is odd and $m \leq d(d-3)$. Using $\mu = d(d-3) + 1$, the first inequality is now immediate from Theorem 2.2(b), since $\mu < n - 1$, so again there is no integer m in the range $1 \leq m < \mu/(n-1)$. For the second inequality it is enough to see that $d(d-3) + 1 \geq n - 5\sqrt{n} + 1$. But $\Delta \leq 2d \leq 2\sqrt{n}$, so $d^2 = n - \Delta \geq n - 2\sqrt{n}$ and hence $d(d-3) = d^2 - 3d \geq n - 5\sqrt{n}$.

(d) This argument is similar to (c), using Corollary 4.1(b) [HR1] (which says that $\alpha(m^{[n]}) \geq m\sqrt{n}$ for $10 \leq n$ when Δ is even and $m \leq d(d-3)/2$) using $\mu = d(d-3)/2 + 1$.

(e) and (f): These require a more delicate analysis than what was given in [HR1]. One instead uses the approach of [HR1] to study α for sequences of multiplicities $m_1, \ldots, m_n$ for which the m_i are not equal. Since the analysis is somewhat arduous and may not be of interest to all readers, we have moved the details to the appendix. This approach can also be used to obtain minor improvements for the results of parts (c) and (d) above; see version 2 of this preprint at arXiv:math/0309064v2. $\qquad\square$

3. The Algorithm

It is worth emphasizing that the bounds presented in Theorem 2.1 are obtained by making simplifying estimates. For specific values of n, we can obtain even better results by applying our algorithm directly using the results of [HR1], as we will demonstrate in this section. In particular, we give bounds for specific values of n in Table 2. Except for $n = 41$ [H] and $n = 17, 19, 22, 26, 37, 50, 65$ and 82 [Bi] for which Table 2 shows previously known bounds that are as good or better than what we can obtain, the results shown in Table 2 are new and better than what was known previously.

We now describe in more detail the conceptual basis for our approach. First produce a set S_n of classes that contains every possible $F(0)$-abnormal class. This we can do using Lemma 3.1 and Proposition 3.2. Clearly, S_n is the union for all $\delta > 0$ of the sets $S_n(\delta)$, where $S_n(\delta)$ is the set of all $H \in S_n$ such that $F(\delta) \cdot H < 0$. The sets $S_n(\delta)$ form a nested sequence of sets that become larger as δ decreases, with the property that if $C \in S_n$ but $C \notin S_n(\delta)$, then $C \cdot F(\delta) \geq 0$.

By Lemma 3.1, each set $S_n(\delta)$ is finite. (Since nefness of classes of positive self-intersection is Zariski open, it is enough in Lemma 3.1 and Proposition 3.2 to require the points p_i to be general.) If for some $\delta > 0$ we can somehow show that $S_n(\delta)$ does not contain an $F(\delta)$-abnormal class, either directly or by

showing that $F(\delta)$ is nef, then it follows that $\varepsilon(n) > 1/\sqrt{n+\delta}$. Even better, it follows that $\varepsilon(n) \geq t$, where t is the minimum ratio $d/(m_1 + \cdots + m_n)$ among all classes $dL - m_1E_1 - \cdots - m_nE_n \in S_n$ with $F(\delta) \cdot (dL - m_1E_1 - \cdots - m_nE_n) \geq 0$.

The approach we take here for attempting to show that $S_n(\delta)$ does not contain an $F(\delta)$-abnormal class is to show that the classes in $S_n(\delta)$ are not the classes of effective divisors. For this we use an intersection theoretic algorithm developed in [HR1] for obtaining lower bounds for the least degree α of curves passing through given points with given multiplicities. If $dL - m_1E_1 - \cdots - m_nE_n \in S_n(\delta)$ but d is less than the lower bound obtained for $\alpha(m_1, \cdots, m_n)$ from [HR1], then $dL - m_1E_1 - \cdots - m_nE_n$ is not an $F(\delta)$-abnormal class. If in this way we show that no element of $S_n(\delta)$ is $F(\delta)$-abnormal, then as above we obtain a lower bound for $\varepsilon(n)$, and at the same time we conclude that $F(\delta)$ is nef. Thus our approach is in fact also a method for verifying nefness.

The following result is a restatement for $\mathbf{P}^2$ of Lemma 2.1.4 [HR2].

Lemma 3.1. *Let X be the blow up of general points $p_1, \ldots, p_n \in \mathbf{P}^2$. Let $\delta > 0$. If H is an $F(\delta)$-abnormal class, then $H = tL - h_1E_1 - \cdots - h_nE_n$ for some non-negative integers $h_1, \ldots, h_n$ and d such that:*
(a) $h_1^2 + \cdots + h_n^2 < (1 + n/\delta)^2/\gamma$, where γ is the number of nonzero coefficients $h_1, \ldots, h_n$, and
(b) $h_1^2 + \cdots + h_n^2 - a \leq t^2 < (l_1h_1 + \cdots + l_nh_n)^2/(n + \delta)$, where a is the minimum positive element of $\{h_1, \ldots, h_n\}$.

The sets $S_n(\delta)$ which would be obtained by applying Lemma 3.1 are much larger than necessary. The following result (which is just a special case of Corollary 2.2.2 [HR2]) subsumes Lemma 3.1 but is much more restrictive, hence it gives much smaller sets $S_n(\delta)$ of prospective abnormal classes. (Note that Corollary 2.2.2(b) [HR2] allows the possibility that $m = -k = 1$. But this can happen only if a class abnormal for n is also abnormal for $n - 1$. This is indeed possible in general: $L - E_1 - E_2$ is abnormal for both $n = 2$ and $n = 3$. However, it follows from [ST] that this does not happen when $n \geq 10$.)

Proposition 3.2. *Let X be obtained by blowing up $n \geq 10$ general points $p_1, \ldots, p_n \in \mathbf{P}^2$. If H is the class of an $F(0)$-abnormal curve, then there are integers $t > 0$, $m > 0$, k and $1 \leq i \leq n$ such that:*
(a) $H = tL - m(E_1 + \cdots + E_n) - kE_i$;
(b) $-m < k$, $k^2 < (n/(n-1)) \min(m, m+k)$;
(c) $m^2n + 2mk + \max(k^2 - m, k^2 - (m+k), 0) \leq t^2 < m^2n + 2mk + k^2/n$ when $k^2 > 0$, but $m^2n - m \leq t^2 < m^2n$ when $k = 0$; and
(d) $t^2 - (m+k)^2 - (n-1)m^2 - 3t + mn + k \geq -2$.

Our algorithm also uses the following result, which is just a special case of Corollary 2.2.5 [HR2]. Note that $\delta = (\mu - 1/n)^{-1}$ is equivalent to $1/\sqrt{n+\delta} = (1/\sqrt{n})\sqrt{1 - 1/(\mu n)}$. Thus $\varepsilon(n) \geq (1/\sqrt{n})\sqrt{1 - 1/(\mu n)}$ is equivalent to the statement that $F(\delta)$ is nef for $\delta = (\mu - 1/n)^{-1}$.

Proposition 3.3. *Let X be obtained by blowing up $n \geq 10$ general points of $\mathbf{P}^2$. Let $\mu \geq 1$ be real and consider the $\mathbf{R}$-divisor class $F(\delta) = \sqrt{n+\delta}L - (E_1 + \cdots + E_n)$, where $\delta = (\mu - 1/n)^{-1}$. Then any $F(\delta)$-abnormal class is of the form $C(t, m, k)$, where t, m and k are as in 3.2 and where $0 < m < \mu$ and either $k = 0$ or $m(n-1) < \mu$.*

We now demonstrate our approach. To verify $\varepsilon(n) \geq (1/\sqrt{n})\sqrt{1 - 1/(\mu n)}$ for some choice of $\mu > 1$, make a list of all (t, m, k) satisfying the criteria of Proposition 3.3. If for each class $C(t, m, k)$ either $F(\delta) \cdot C(t, m, k) \geq 0$ or, by the results of [HR1], $C(t, m, k)$ is not the class of an effective divisor, then $\varepsilon(n) \geq (1/\sqrt{n})\sqrt{1 - 1/(\mu n)}$.

In practice, of course, one does not know ahead of time what μ to pick, so one finds all triples (t, m, k) satisfying Proposition 3.3, starting with $m = 1$, and successively increasing m. Call such a triple a *candidate* triple. For each candidate triple, compute $e(t, m, k)$, where we define $e(t, m, k)$ by $(1/\sqrt{n})\sqrt{1 - 1/(e(t, m, k)n)} = t/(mn + k)$ (equivalently, such that $F(\delta') \cdot C(t, m, k) = 0$, where $\delta' = (e(t, m, k) - 1/n)^{-1}$). If for some candidate triple (t, m, k), $e(t, m, k)$ is such that for each candidate triple (t', m', k') with $m' < e(t, m, k)$ (if $k = 0$) or $m' < e(t, m, k)/(n-1)$ (if $k \neq 0$) we have either $e(t', m', k') \geq e(t, m, k)$ or we can show that $C(t', m', k')$ is not the class of an effective, reduced, irreducible divisor (and hence not an abnormal class), then we obtain the bound $\varepsilon(n) \geq (1/\sqrt{n})\sqrt{1 - 1/(e(t, m, k)n)}$.

We now carry this out for $n = 10$. Here is a list of all triples (t, m, k) satisfying Proposition 3.2 for $n = 10$, with $m \leq 182$:

t	m	k	e	t	m	k	e	t	m	k	e	t	m	k	e
3	1	0	1	117	37	0	1369	234	74	0	1369	430	136	0	308.26
6	2	-1	36.1	154	49	-3	2635.21	308	98	-6	2635.21	449	142	0	517.02
22	7	0	8.16	177	56	0	101.16	313	99	0	239.04	456	144	2	51984.1
41	13	0	18.77	191	60	4	6080.26	332	105	0	424.03	456	145	-8	51984.1
60	19	0	36.1	191	61	-6	6080.26	351	111	0	1369	468	148	0	1369
79	25	0	69.44	196	62	0	160.16	382	120	8	6080.26	547	173	0	369.49
80	25	3	711.21	215	68	0	308.26	419	132	5	11704.16	566	179	0	593.35
98	31	0	160.16	228	72	1	51984.1	419	133	-5	11704.16	573	182	-8	6080.26

TABLE 1. All Proposition 3.2 test classes $C(t, m, k)$ with $m \leq 182$ and corresponding values $e(t, m, k)$.

It is easy to see that neither $C(3, 1, 0)$ nor $C(6, 2, -1)$ can be the class of an effective divisor (use the fact that there is a unique plane curve of degree $3m$ with 9 general points of multiplicity m). By [CCMO], no abnormal curve occurs with $n \geq 10$, $m \leq 20$ and $k = 0$, which rules out $C(22, 7, 0)$, $C(41, 13, 0)$ and $C(60, 19, 0)$. We would like to thank Professor Rick Miranda, who (in a personal communication) shared with us his proof that $C(79, 25, 0)$ is not the class of an effective divisor, using the method of [CM2] (which is a refinement of [CM1]). Note that for $e(177, 56, 0) = 101.16$ there is no triple (t', m', k') such that both $m' < e(177, 56, 0)$ and $C(t', m', k')$ is effective. Thus $F = (560/177)L - (E_1 + \cdots + E_{10})$ is nef, $F \cdot C(177, 56, 0) = 0$, and we have $\varepsilon(10) \geq 177/560 = (1/\sqrt{10})\sqrt{1 - 1/(10e(177, 56, 0))}$. To improve on this bound, we would need to show that $C(177, 56, 0)$ is not the class of a reduced, irreducible curve (it suffices, of course, to show it is the class of no effective divisor). Were we able to do this, we next would need to deal with $C(98, 31, 0)$ and $C(196, 62, 0)$, and so on.

Determining that $C(t, m, k)$ is *not* the class of an effective, reduced, irreducible divisor is a task that in principle can be done computationally, with the only restrictions imposed by the computational resources available. Thus the method just explained can be used to get bounds on $\varepsilon(n)$ arbitrarily close to $1/\sqrt{n}$ if Nagata's conjecture is true, and would eventually lead to a counterexample and an exact value of $\varepsilon(n)$ if it were false.

We close with a list of the best currently known values of $f(n)$ when n is not square, for $10 \leq n \leq 99$.

n	f	C(t,m,0)	n	f	C(t,m,0)	n	f	C(t,m,0)
10	1011.61	C(177,56)	41	1025	C(160,25)	71	6819.08	C(792,94)
11	402.28	C(106,32)	42	1306.94	C(149,23)	72	3008.34	C(263,31)
12	300.52	C(83,24)	43	1741.5	C(236,36)	73	8129.89	C(786,92)
13	325	C(90,25)	44	1985.5	C(252,38)	74	9085.64	C(929,108)
14	740.6	C(86,23)	45	3782.25	C(275,41)	75	9409	C(840,97)
15	566.78	C(89,23)	46	3140.26	C(217,32)	76	5337.1	C(462,53)
17	1089	C(136,33)	47	7109.17	C(994,145)	77	13862.75	C(1246,142)
18	466.94	C(89,21)	48	1521.39	C(187,27)	78	5698.52	C(627,71)
19	28900	C(170.39)	50	9801	C(700,99)	79	19525.09	C(2142,241)
20	660.64	C(143,32)	51	3313.98	C(407,57)	80	5107.27	C(474,53)
21	1187.1	C(142,31)	52	6257.33	C(274,38)	82	26569	C(1476,163)
22	38809	C(197,42)	53	3499.89	C(313,43)	83	8381.98	C(829,91)
23	576	C(115,24)	54	5713.2	C(338,46)	84	7709.47	C(724,79)
24	1009.2	C(142,29)	55	2370.64	C(304,41)	85	5802.66	C(295,32)
26	2601	C(260,51)	56	3193.01	C(419,56)	86	14198.76	C(1493,161)
27	997.96	C(161,31)	57	2608.42	C(234,31)	87	5497.02	C(457,49)
28	1304.25	C(201,38)	58	9802	C(396,52)	88	8530.92	C(666,71)
29	639.45	C(113,21)	59	3352.27	C(192,25)	89	7281.81	C(566,60)
30	1230.76	C(219,40)	60	7562.5	C(852,110)	90	13690	C(702,74)
31	1093.26	C(128,23)	61	5380.2	C(328,42)	91	5126.33	C(372,39)
32	940.52	C(147,26)	62	12164.13	C(1496,190)	92	13370.32	C(1103,115)
33	1093.55	C(178,31)	63	2242.33	C(246,31)	93	6076	C(405,42)
34	1731.93	C(239,41)	65	16641	C(1040,129)	94	14950.51	C(1367,141)

35	974.47	C(136,23)	66	5410.98	C(593,73)	95	6390.76	C(614,63)
37	5329	C(444,73)	67	5550.49	C(532,65)	96	18070.33	C(1695,173)
38	1898.97	C(265,43)	68	4442.13	C(437,53)	97	4773.3	C(453,46)
39	1779.7	C(231,37)	69	8283.45	C(407,49)	98	29804.08	C(2950,298)
40	1601.66	C(196,31)	70	5603.33	C(343,41)	99	6892.38	C(587,59)

TABLE 2. Best currently known values of $f(n)$ for nonsquares $10 \leq n \leq 99$.

For each n, Table 2 gives the best value we know for $f(n)$ (truncated to two decimals), along with a possible abnormal curve $C(t, m, k)$ which we are unable to rule out but which would have to be ruled out in order to verify a larger value for $f(n)$. Thus the bound on $\varepsilon(n)$ we obtain for each n is $\varepsilon(n) \geq (1/\sqrt{n})(\sqrt{1 - 1/f(n)})$, and if there actually is an abnormal class $C(t, m, k)$, then we would have equality. It turns out that $k = 0$ for each of the cases listed, so we write $C(t, m)$ in place of $C(t, m, 0)$. That k should be 0 in these examples is not too surprising, since by Proposition 3.3 it follows that the constraints for the occurrence of an abnormal curve with $k \neq 0$ are much more severe. Thus, as Table 1 suggests, cases with $k \neq 0$ do not come into play until one is trying to verify $\varepsilon(n) \geq (1/\sqrt{n})(\sqrt{1 - 1/(\mu n)})$ for values of μ that are quite large.

The bound given for $n = 41$ is due to [H] (our method also obtains this value, but does not improve on it). For $n = 19$, $n = 22$ and $n > 10$ such that $n - 1$ is a square, the bounds given here are due to [Bi]. Except in these cases, the listed values come from applying the method discussed above, using the intersection theoretic algorithm of [HR1] when necessary to show $C(t, m, k)$ cannot be the class of an effective divisor. (The [HR1] algorithm depends on two parameters r and d which can be chosen somewhat arbitrarily. For Table 2, we used $d = \lfloor \sqrt{n} \rfloor$ and $r = \lfloor d\sqrt{n} \rfloor$. The listed classes $C(t, m, k)$ are just ones with $F(\delta) \cdot C(t, m, k) = 0$ but which the [HR1] bound on effectivity is not good enough to rule out, thereby preventing us from obtaining a larger value for $f(n)$. It is possible that by employing other choices for r and d we could improve some of the bounds even further.)

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A. APPENDIX

Whereas Theorem 2.2 depends only on the intersection theoretic considerations of [HR2], further improvements, which we will need in order to prove Theorem 2.1 (e, f), are possible based on more delicate geometric considerations involving curves. As an example, we have:

Theorem A.1. *Let $n \geq 10$ and $\mu > 0$ be integers. Define $d = \lfloor \sqrt{n} \rfloor$, $g = (d-1)(d-2)/2$, $r = \lfloor d\sqrt{n} \rfloor$ and $\nu = (\mu - 1)/(n - 1)$. Assume either that $\mu \leq 6(n-1)$, or that $\mu \leq n(n-1)$ and*

$$\frac{\nu r + g - 1}{d} - 1 \geq \left(\nu - \frac{d}{n} \right) \sqrt{n - \frac{1}{\mu}}.$$

If $\alpha_0(m^{[n]}) \geq m\sqrt{n - \frac{1}{\mu}}$ for every integer $1 \leq m < \mu$, then $\varepsilon(n) \geq (1/\sqrt{n})\sqrt{1 - 1/(n\mu)}$.

In this appendix, we apply Theorem A.1 to prove the following result, from which we will obtain the explicit bounds given in Theorem 2.1(e, f):

Lemma A.2. *Let $1 \leq \mu \leq n(n-1)$ be integers with $n \geq 10$, and define $d = \lfloor \sqrt{n} \rfloor$, $g = (d-1)(d-2)/2$ and $r = \lfloor d\sqrt{n} \rfloor$. Assume that*

$$\frac{(\mu - 1)r + g - 1}{d} \geq (\mu - 1)\sqrt{n - \frac{1}{\mu}}.$$

Then $\varepsilon(n) \geq (1/\sqrt{n})\sqrt{1 - 1/(\mu n)}$.

In order to apply Theorem 2.2, we need to verify certain lower bounds on minimum degrees α of curves with points of given multiplicities. A means of deriving such bounds is given in [HR1]. Indeed, as pointed out in Remark 1.2, bounds given in [HR1] in the case of uniform multiplicities already imply $\varepsilon(n) \geq (1/\sqrt{n})\sqrt{1 - 1/(n(n - 5\sqrt{n})/2)}$ for $n \geq 10$. The main point of this appendix is to analyze the method of [HR1] to obtain explicit bounds (given in Theorem A.6) when the multiplicities are only almost uniform, which we then use to obtain the improved bounds on $\varepsilon(n)$ given in Theorem A.1, Lemma A.2 and Theorem 2.1.

The approach developed in [HR1] for obtaining lower bounds for the least degree $\alpha(\mathbf{m})$ of a curve with multiplicities $\mathbf{m} = (m_1, \ldots, m_n)$ at a set of n general points depends on choosing arbitrary positive integers $r \leq n$ and d, and then involves specializing the n points and using semicontinuity. The specialization consists in choosing first an irreducible plane curve C of degree d, and then choosing points $p_1, \ldots, p_n$ in the following way. We will denote by X_i the surface obtained from $X_0 = \mathbf{P}^2$ by blowing up, in order, the points $p_1, \ldots, p_i$, where p_1 is a general smooth point of $C \subset X_0$; p_i is infinitely near p_{i-1} for $2 \leq i \leq n$; and p_i is a point of the proper transform of C on X_{i-1} for $i \leq r$ (more precisely, so that $E_i - E_{i+1}$ is the class of an effective, reduced and irreducible divisor for $0 < i < n$ and so that the class of the proper transform of C to X is $dL - E_1 - \cdots - E_r$). Denoting by $\alpha'(\mathbf{m})$ the least degree t such that $|tL - m_1 E_1 - \cdots - m_n E_n|$ is non-empty (for this special position of the points) we have $\alpha(\mathbf{m}) \geq \alpha'(\mathbf{m})$ by semicontinuity. Now [HR1] gives a numerical algorithmic criterion for $h^0(X, \mathcal{O}_X(t_0 L - m_1 E_1 - \cdots - m_n E_n))$ to vanish. If t satisfies the criterion (and hence $h^0 = 0$), then $\alpha'(\mathbf{m}) > t$. The largest t which satisfies the criterion is our lower bound.

To describe the criterion, we recall some notation from [HR1]. Given a class $D_0 = t_0 L - m_1 E_1 - \cdots - m_n E_n$ such that $m_1 \geq \cdots \geq m_n \geq 0$ and given $[C] = dL - E_1 - \cdots - E_r$ as above, we define classes D_i' and D_i for $i \geq 0$. First, $D_i' = D_i - [C]$. Then D_{i+1} is obtained from D_i' by *unloading*; i.e., let $F = D_i'$, let $N_j = E_j - E_{j+1}$ for $1 \leq j < n$ and let $N_n = E_n$. Whenever $F \cdot N_j < 0$, replace F by $F - N_j$. Eventually it happens that $F \cdot N_j \geq 0$ for all j, in which case we set D_{i+1} equal to the resulting F. (Since under the specialization each N_j is the class of a reduced irreducible divisor, in the event that D_i' is the class of an effective divisor, unloading just amounts to subtracting off certain fixed components of $|D_i'|$. Although it is convenient to define D_i for all i, we are only interested in D_i when i is reasonably small. Indeed, for i sufficiently large, D_i always takes the form of a negative multiple of L; the multiplicities all eventually unload to 0. In fact, when D_0 is understood, we will denote by ω' the least i such that $D_i \cdot E_j = 0$ for all $j > 0$.)

Denote $D_i \cdot L$ by t_i. Let j be the least index i such that $t_i < d$ and let $g_C = (d-1)(d-2)/2$ be the genus of C. The criterion of [HR1] (see the discussion after the proof of Lemma 2.3 of [HR1]) is that

$$\text{if } D_i \cdot C \leq g_C - 1 \text{ for } 0 \leq i < j \text{ and } (t_j + 1)(t_j + 2) \leq 2(dt_j - D_j \cdot C) \text{ then } \alpha'(\mathbf{m}) > t_0. \qquad (**)$$

The results of [HR1] are obtained by analyzing this criterion with respect to particular choices of the parameters d and r describing C.

The results we obtain here mostly involve choosing $d = \lfloor\sqrt{n}\rfloor$ and $r = \lfloor d\sqrt{n}\rfloor$, however other choices can also be useful; for instance, the case $\Delta = 2$ in Theorem 2.1 follows from a computation where $r = \lceil d\sqrt{n}\rceil$ is used.

We will find it useful to have a refinement of Proposition 3.2 in the case that $m < n$:

Lemma A.3. *Let X be obtained by blowing up $n \geq 10$ general points $p_1,\ldots,p_n \in \mathbf{P}^2$. Assume $[C] = tL - (m + k)E_1 - mE_2 - \cdots - mE_n$ is the class of an almost uniform abnormal curve C with $n > m > 0$. Then $m + k > 0$ and $-\sqrt{m} \leq k \leq \sqrt{m}$. Moreover, if $k \neq 0$, then also $2mk = t^2 - m^2n$ (or equivalently $C^2 = -k^2$) and $m\sqrt{n} - 1 < t < m\sqrt{n} + 1$.*

Proof. To see $-\sqrt{m} \leq k \leq \sqrt{m}$, observe that $m < n$ implies $mn/(n-1) \leq m+1$; now apply $k^2 < mn/(n-1)$ from Proposition 3.2(b). Note that Proposition 3.2(b) also gives $m + k > 0$.

Now, assume that $k \neq 0$. By Proposition 3.2(c) we have $t^2 - nm^2 - 2mk < k^2/n$, but now $k^2/n < 1$; Proposition 3.2(c) also tells us that $t^2 - nm^2 - 2mk \geq 0$. Therefore, putting both inequalities together we must have $t^2 - nm^2 - 2mk = 0$, proving $2mk = t^2 - m^2n$ and thus $C^2 = -k^2$.

Finally, as C is abnormal, we have $t < m\sqrt{n} + k/\sqrt{n} < m\sqrt{n} + 1$. On the other hand, since $-k^2 = C^2 \geq -(m + k)$ by [X1], we have $(k - 1/2)^2 \leq m + 1/4 < n$, so $k > 1/2 - \sqrt{n}$ and $t = \sqrt{m^2n + 2mk} > \sqrt{m^2n - 2m(\sqrt{n} - 1/2)} \geq \sqrt{(m\sqrt{n} - 1)^2}$, and we conclude $t > m\sqrt{n} - 1$. $\square$

Remark A.4. We note that when Lemma A.3 applies, there is for each m at most one $k \neq 0$ and one t for which an abnormal curve $[C] = tL - (m + k)E_1 - mE_2 - \cdots - mE_n$ could exist. Indeed, $2mk = t^2 - m^2n$ implies that t^2 has the same parity as m^2n, and only one integer t in the range $m\sqrt{n} - 1 < t < m\sqrt{n} + 1$ has this property.

From now on we restrict our attention to almost uniform sequences $\mathbf{m} = (m + k, m, \ldots, m)$ of n multiplicities satisfying the inequalities imposed by Proposition 3.2 or Lemma A.3. To apply the criterion of [HR1], the multiplicities in $\mathbf{m}$ should be nonincreasing. Thus we will assume $\mathbf{m} = (m + k, m^{[n-1]})$ when $k \geq 0$ and $\mathbf{m} = (m^{[n-1]}, m + k)$ when $k \leq 0$. In the special case that $m < n$ and $k \geq 0$, we have $k^2 \leq m$ by Lemma A.3, in which case we let $\mathbf{m}'$ denote $((m + 1)^{[k]}, m^{[n-k]})$. (In the terminology of [HR1], $\mathbf{m}'$ is then n-semiuniform.) If $m < n$ but $k < 0$, we take $\mathbf{m}' = \mathbf{m}$. Since after the specialization of [HR1], $E_i - E_{i+1}$ is the class of an effective divisor for each $i > 0$, clearly $\alpha'(\mathbf{m}) \geq \alpha'(\mathbf{m}')$, so a lower bound for $\alpha'(\mathbf{m}')$ is also a lower bound for $\alpha'(\mathbf{m})$ and hence for $\alpha(\mathbf{m})$.

Lemma A.5. *Let n be a positive integer. Let $d = \lfloor\sqrt{n}\rfloor$, $r = \lfloor d\sqrt{n}\rfloor$, and assume $[C]$ as above is $dL - E_1 - \cdots - E_r$ and $0 \leq k^2 \leq n$. Let $D_0 = tL - (m + 1)E_1 - \cdots - (m + 1)E_k - mE_{k+1} - \cdots - mE_n$ if $k \geq 0$, or $D_0 = tL - mE_1 - \cdots - mE_{n-1} - (m + k)E_n$ if $k < 0$, and let ω' be the least $i \geq 0$ such that $D_i \cdot E_j = 0$ for all $j > 0$. Then $dt - (mr + k) \geq D_i \cdot C$ for all $0 \leq i < \omega'$. Moreover, if $k < 0$ and $\Delta = n - d^2$ is even and positive, then $dt - mr \geq D_i \cdot C$ for all $0 \leq i < \omega'$.*

Proof. The proof is similar to that of Lemma 2.3 of [HR1]. Also, it is obviously true if n is a square, since then $C^2 = 0$, so we may assume that n is not a square. Thus $\Delta = n - d^2$ is positive. We begin with some useful observations. If Δ is even, then for some $1 \leq \delta \leq d$ we can write $n = d^2 + 2\delta$, in which case it is not hard to check that $r = d^2 + \delta - 1$. If $\delta = d$, then $n - r - 1 \leq d$ and $d(n - r)/n = d(d + 1)/n < 1$, while if $\delta < d$, then $n \leq d^2 + \delta - 1 + d = r + d$, so again $d(n - r)/n \leq d^2/n < 1$. If Δ is odd, we have $n = d^2 + 2\delta + 1$ with $\delta < d$, and $r = d^2 + \delta$, so again $n \leq r + d$ and $d(n - r)/n < 1$. Thus we always have $d(n - r)/n < 1$ and $n - r - 1 \leq d$.

Now assume that $k \geq 0$. The choice $r = \lfloor d\sqrt{n}\rfloor$ ensures that $r^2/n - d^2 \leq 0$, while $k^2 \leq n$ implies that $\min(k, r) = k$ and $\min(k, r) - kr/n = k(n - r)/n \leq d(n - r)/n < 1$. On the other hand, $D_0 \cdot C = dt - (mr + k)$; thus it is enough to show that $(D_i - D_0) \cdot C \leq i(r^2/n - d^2) + \min(k, r) - kr/n$. Let $A_0 = 0$, and for $0 < j \leq n$ let $A_j = -E_1 - \cdots - E_j$. For $0 \leq i < \omega'$, one can check that $D_i = (t - id)L - (m - i + q)E_1 - \cdots - (m - i + q)E_n + A_\rho$, where $k + i(n - r) = qn + \rho$ and $0 \leq \rho < n$. (To see this, note by construction that D_i always must have the form $(t - id)L - b(E_1 + \cdots + E_n) + A_c$ for some b and c. To determine b and c, use the fact that ω' is such that for $i < \omega'$, the sum of the coefficients of the E_j in D_i is just the sum of the coefficients of the E_j in $D_0 - iC$, hence $bn + c = mn + k - ir$.) It now follows that

$$D_i \cdot C - D_0 \cdot C = i(r - d^2) - rq + A_\rho \cdot C + \min(r, k).$$

The claim now follows using $A_\rho \cdot C = -\min(\rho, r)$ and $(k + i(n-r))(r/n) = (r/n)(qn + \rho) \leq rq + \min(\rho, r)$.

Assume now that $k < 0$. Note that $n - r - 1 \leq d \leq \sqrt{n}$. Also, $|k| \leq \lfloor \sqrt{n} \rfloor = d$, so $|k|(n - r - 1) < d\sqrt{n}$, hence $|k|(n - r - 1) \leq r \leq n - 1$. For $1 \leq i \leq |k|$, one can argue in a way similar to that used before (noting for $i \leq |k|$ that the coefficient of E_n is unaffected), to see that $D_i = (t - id)L - (m + 1 - i)E_1 - \cdots - (m + 1 - i)E_{i(n-1-r)} - (m - i)E_{i(n-1-r)+1} - \cdots - (m - i)E_{n-1} - (m + k)E_n$. Thus $D_i \cdot C = (t - id)d - rm - i(n-1) + 2ri$, hence $(D_{i-1} - D_i) \cdot C = d^2 + n - 2r - 1 > (\sqrt{n} - d)^2 - 1 > -1$. Thus $D_0 \cdot C \geq D_1 \cdot C \geq D_2 \cdot C \geq \cdots \geq D_{|k|} \cdot C$.

Note that $D_{|k|} = (t - |k|d)L - (m - |k|)E_1 - \cdots - (m - |k|)E_n + A_\rho$, where $|k|(n - r - 1) = \rho$. So for $|k| \leq i < \omega'$, we are in a situation similar to that above: we have $D_i = (t - id)L - (m - i + q)E_1 - \cdots - (m - i + q)E_n + A_\rho$, where $k + i(n-r) = qn + \rho$ and $0 \leq \rho < n$, and an analogous argument shows $(D_i - D_0) \cdot C = i(r - d^2) - rq + A_\rho \cdot C \leq i(r^2/n - d^2) - kr/n$. Thus $D_i \cdot C \leq D_0 \cdot C - kr/n \leq D_0 \cdot C + |k| = dt - (mr + k)$, as we wished to show.

Finally, suppose $\Delta = n - d^2$ is even. As noted above, we can write $n = d^2 + 2\delta$ and $r = d^2 + \delta - 1$, hence $n = 2r - d^2 + 2$. Using this expression for n we have $(i(r^2 - d^2n) - kr)/n = (i((r - d^2)^2 - 2d^2) - kr)/n$, and using $-k < i$, we have $(i((r - d^2)^2 - 2d^2) - kr)/n \leq (i((r - d^2)^2 - d^2) + k(d^2 - r))/n = (i/n)(\delta^2 - \delta - d^2) + (\delta - 1)(|k| - i)/n < 0$. Thus $dt - mr = D_0 \cdot C \geq D_i \cdot C$ for $0 \leq i < \omega'$. □

The following theorem extends Theorem 1.3 of [HR1] to almost uniform classes for our particular choice of r and d. Given a multiplicity sequence $\mathbf{m} = (m_1, \ldots, m_n)$, define u and ρ by: $u \geq 0$, $0 < \rho \leq r$ and $m_1 + \cdots + m_n = ur + \rho$.

Theorem A.6. *Given an integer n, let $d = \lfloor \sqrt{n} \rfloor$, $r = \lfloor d\sqrt{n} \rfloor$ and $\mathbf{m} = (m, \ldots, m, m + k)$, with $k^2 \leq m < n$. Define u and ρ as above, denote the genus $(d-1)(d-2)/2$ of a plane curve of degree d by g and let s be the largest integer such that we have both $(s+1)(s+2) \leq 2\rho$ and $0 \leq s < d$. Then*

$$\alpha(\mathbf{m}) \geq 1 + \min(\lfloor (mr + k + g - 1)/d \rfloor, s + ud).$$

Moreover, if $k < 0$ and $\Delta = n - d^2$ is even and positive, then $\alpha(\mathbf{m}) \geq 1 + \min(\lfloor (mr + g - 1)/d \rfloor, s + ud)$.

Proof. As discussed above, we may replace $\mathbf{m}$ by $\mathbf{m}'$, hence we may assume $\mathbf{m} = (m, m, \ldots, m, m + k)$ if $k \leq 0$, and $\mathbf{m} = (m + 1, \ldots, m + 1, m, \ldots, m)$ if $k > 0$. In either case, we define m_i by $(m_1, \ldots, m_n) = \mathbf{m}$, and let $D_0 = tL - (m_1E_1 + \cdots + m_nE_n)$. Our aim is to show that if $t \leq \min(\lfloor (mr + k + g - 1)/d \rfloor, s + ud)$ (or $t \leq \min(\lfloor (mr + g - 1)/d \rfloor, s + ud)$ in case $k < 0$ and Δ is even and positive), then, as in (**), $D_i \cdot C \leq g - 1$ for $0 \leq i < j$ and $(t_j + 1)(t_j + 2) \leq 2(dt_j - D_j \cdot C)$, where j is the least index i such that $t_i < d$.

It is easy to check that ω', defined above, is $\lceil (mn + k)/r \rceil = u + 1$, so if $t \leq s + ud$, it follows that $t_{\omega'} \leq s - d < 0$, and thus $\omega' \geq \omega$, where ω is the least i such that $t_i < 0$, and hence $\omega = j + 1$. Lemma A.5 now gives $D_i \cdot C \leq dt - (mr + k)$ (resp., $D_i \cdot C \leq dt - mr$, if $0 < \Delta$ is even and $k < 0$) for all $0 \leq i \leq \omega - 2$, so $t \leq \lfloor (mr + k + g - 1)/d \rfloor$ (resp., $t \leq \lfloor (mr + g - 1)/d \rfloor$) implies $D_i \cdot C \leq g - 1$. To conclude that $\alpha'(\mathbf{m}) \geq t + 1$, it is now enough to check that $(t - jd + 1)(t - jd + 2) \leq 2v_j$, where for any i we define $v_i = dt_i - D_i \cdot C$.

If $j = u$ (i.e., $\omega' = \omega$), we have $\rho = v_j$. But $t - jd = t - ud \leq s$, hence $(t - jd + 1)(t - jd + 2) \leq (s + 1)(s + 2) \leq 2\rho = 2v_j$. If $j < u$ (so $\omega' > \omega$), by definition of j we at least have $t - jd \leq d - 1$, so $(t - jd + 1)(t - jd + 2) \leq d(d + 1)$. But $\omega' > \omega$ implies $v_j > r$, and $r \geq d^2$ implies $d(d + 1) \leq 2r$, so again $2v_j \geq 2r \geq d(d + 1) \geq (t - jd + 1)(t - jd + 2)$. □

We can now prove Theorem A.1 and Lemma A.2.

Proof of Theorem A.1. By Theorem 2.2, it is enough to prove that if $1 \leq m < \mu/(n-1)$, $0 < k^2 < (n/(n-1))\min(m, m + k)$ and μ satisfies the hypotheses then $\alpha_0(m, \ldots, m, m + k) \geq ((mn + k)/\sqrt{n})\sqrt{1 - 1/(n\mu)}$. In the case that $\mu \leq 6(n - 1)$, the only multiplicities involved are $m \leq 5$, and then the claim follows by [LU]. So assume $\mu \leq n(n - 1)$ and

$$\frac{\nu r + g - 1}{d} - 1 \geq \left(\nu - \frac{d}{n}\right)\sqrt{n - \frac{1}{\mu}} \qquad (***).$$

The conclusion is true when n is a square, so we may assume n is a nonsquare. Cases $10 \leq n < 25$ (i.e., $3 \leq d \leq 4$) we treat ad hoc, briefly. When $d = 3$, it turns out that the only values of μ satisfying the hypotheses have $\mu \leq 6(n - 1)$, and so were already dealt with. For $d = 4$, the same is true except for $n = 23$, since $133 \leq \mu \leq 163$ satisfies (***) and has $\mu > 6(n - 1)$. The resulting values of m are 6 and 7, and k must by Proposition 3.2(b) be ± 1 or ± 2. It is easy to check that in these cases there is no square t^2 meeting the

conditions of Proposition 3.2(c), which means $\alpha_0(m, \ldots, m, m + k) \geq ((mn + k)/\sqrt{n})\sqrt{1 - 1/(n\mu)}$. So the claim holds for $n < 25$ and hereafter we may assume that $d \geq 5$.

The condition $\mu \leq n(n-1)$ guarantees that $m < n$ and thus we can apply Theorem A.6 to bound $\alpha_0(m, \ldots, m, m + k)$. Thus it is enough to show that $((mn + k)/\sqrt{n})\sqrt{1 - 1/(n\mu)}$ is no bigger than the bound given in Theorem A.6. First we show that $s + ud + 1 \geq (mn + k)/\sqrt{n}$. Since $r^2 \leq d^2 n$, we see that $(mn+k)/\sqrt{n} \leq (mn+k)d/r$, so it suffices to show that $(mn+k)d/r \leq s+ud+1$. If $s = d-1$, then $s+ud+1 = (u+1)d = \lceil (mn+k)/r \rceil d \geq (mn+k)d/r$ as required, so assume $(s+1)(s+2) \leq 2\rho < (s+2)(s+3)$ and $s+2 \leq d$. Then $r(s+ud+1) = r(s+1)+(mn+k)d-d\rho$, so we need only check that $r(s+1)+(mn+k)d-d\rho \geq (mn+k)d$, or $r(s+1) \geq d\rho$. If $s = 0$, then $r(s+1) \geq r \geq 3d = d(s+2)(s+3)/2 \geq d\rho$, since $\sqrt{n} \geq 3$. If $s > 0$, then $r(s+1) \geq d(s+3)(s+2)/2 \geq d\rho$, since $r(s+1)/d \geq d(s+1) \geq (s+2)(s+1) \geq (s+3)(s+2)/2$.

It remains to prove that $\lfloor (mr + k + g - 1)/d \rfloor + 1 \geq ((mn + k)/\sqrt{n})\sqrt{1 - 1/(n\mu)}$. Observe that $\lfloor x/d \rfloor + 1 \geq x/d$, so it is enough to prove $(mr + k + g - 1)/d \geq ((mn + k)/\sqrt{n})\sqrt{1 - 1/(n\mu)}$, which we can rewrite as $m(\sqrt{n - 1/\mu} - r/d) \leq (d - 3)/2 + k(1/d - (1/\sqrt{n})\sqrt{1 - 1/(\mu n)})$. But $k^2 \leq m < n$ by Lemma A.3, so $k \geq -d$, hence it is enough to prove

$$ m\left(\sqrt{n - \frac{1}{\mu}} - \frac{r}{d}\right) \leq \frac{d-3}{2} - \left(1 - \frac{d}{\sqrt{n}}\sqrt{1 - \frac{1}{\mu n}}\right). $$

As $d \geq 5$, the term on the right is positive, so the inequality holds for $m = 0$. But the term on the left is linear in m, so it will suffice to show that the inequality holds for $m = (\mu - 1)/(n - 1) = \nu$, and this is equivalent to (***). $\qquad\square$

Proof of Lemma A.2. The conclusion is true when n is a square, so we may assume n is a nonsquare. As we did in the proof of Theorem A.1, we treat cases $10 \leq n < 25$ (i.e., $3 \leq d \leq 4$) with ad hoc arguments. When $d = 3$, it turns out that the only value of μ satisfying the hypothesis is $\mu = 1$. For $d = 4$, it turns out that μ is never more than 19. From Table 2, we see that $\varepsilon(n) \geq (1/\sqrt{n})\sqrt{1 - 1/(\mu n)}$ thus holds for $n < 25$. So hereafter we may assume that $d \geq 5$.

Theorem A.1 will imply our conclusion. To apply Theorem A.1, first note that μ satisfies the inequalities $\mu \leq n(n-1)$ (by hypothesis) and

$$ \frac{\nu r + g - 1}{d} - 1 \geq \left(\nu - \frac{d}{n}\right)\sqrt{n - \frac{1}{\mu}}. $$

To justify this second inequality, observe that

$$ (\nu - d/n)\sqrt{n - 1/\mu} = (1/(n-1))(\mu - 1)\sqrt{n - 1/\mu} - (d/n)\sqrt{n - 1/\mu} $$

and $(1/(n-1))(\mu - 1)\sqrt{n - 1/\mu} - (d/n)\sqrt{n - 1/\mu} \leq (1/(n-1)((\mu - 1)r + g - 1)/d - (d/n)\sqrt{n - 1/\mu}$ by hypothesis, so it will follow from

$$ \frac{g-1}{d} - 1 \geq \frac{g-1}{d(n-1)} - \frac{d}{n}\sqrt{n - \frac{1}{\mu}}. $$

Substituting $g = (d-1)(d-2)/2$ and rearranging the terms, this is equivalent to $1 \leq (d-3)(n-2)/(2n-2) + (d/n)\sqrt{n - 1/\mu}$. But $d \geq 5$ and $\mu \geq 1$, so $(d-3)(n-2)/(2n-2) + (d/n)\sqrt{n - 1/\mu} \geq (n-2)/n + (5/n)\sqrt{n - 1}$, and it is immediate that the last expression is bounded below by 1 when $n > 25$.

The other hypothesis in Theorem A.1 that needs to be checked is that $\alpha(\mathbf{m}) \geq m\sqrt{n - 1/\mu}$ for uniform multiplicity sequences $\mathbf{m} = (m, \ldots, m)$ in which $m \leq \mu - 1$. To this end we apply Theorem 1.3(c) of [HR1]. What we want is to show that $m\sqrt{n - 1/\mu}$ is no bigger than the lower bound on $\alpha(\mathbf{m})$ given in that theorem. Recall the quantities s, u and ρ defined in Theorem A.6. Exactly as shown in the proof of Corollary 4.1 of [HR1], we have $s + ud + 1 \geq m\sqrt{n}$. We quote: "Since $r^2 \leq d^2 n$, we see that $m\sqrt{n} \leq mnd/r$, so it suffices to show that $mnd/r \leq s + ud + 1$. If $s = d - 1$, then $s + ud + 1 = (u+1)d = \lceil mn/r \rceil d \geq mnd/r$ as required, so assume $(s+1)(s+2) \leq 2\rho < (s+2)(s+3)$ and $s+2 \leq d$. Then $r(s+ud+1) = r(s+1) + mnd - d\rho$, so we need only check that $r(s+1) + mnd - d\rho \geq mnd$, or even that $r(s+1) \geq d(s+2)(s+3)/2$ (which is clear if $s = 0$ since $d \geq 3$) or that $r \geq d^2(s+3)/(2(s+1))$ (which is also clear since now we may assume $s \geq 1$)." Thus, it only remains to prove that $\lfloor (mr + g - 1)/d \rfloor + 1 \geq m\sqrt{n - 1/\mu}$. As in the proof of Theorem A.1, it is enough to prove $(mr + g - 1)/d \geq m\sqrt{n - 1/\mu}$. But both sides of this inequality are linear in m, it obviously holds for $m = 0$, and it holds for $m = \mu - 1$ by hypothesis, so it clearly holds for all $0 < m < \mu$. $\qquad\square$

The next Lemma is a technical result used to prove Theorem 2.1(e, f).

Lemma A.7. *Let $n \geq 17$ be an integer, not a square, and define $d = \lfloor \sqrt{n} \rfloor$, $r = \lfloor d\sqrt{n} \rfloor$, $\Delta = n - d^2$ and $\delta = \lfloor \Delta/2 \rfloor$. Let*

$$\mu_n = \begin{cases} \left\lfloor d\left(d - 3 + \frac{d(d-3)-1}{(d-3)(d^2+\delta+1)}\right) \frac{d^2+\delta}{d^2-\delta^2} \right\rfloor + 1 & \text{if } \Delta = 2\delta + 1 \text{ is odd,} \\ \left\lfloor d\left(d - 3 + \frac{d(d-3)-1}{(d-3)(d^2+\delta)}\right) \frac{d^2+\delta-1}{2d^2-(\delta-1)^2} \right\rfloor + 1 & \text{if } \Delta = 2\delta \text{ is even;} \end{cases}$$

then $\mu = \mu_n$ satisfies the inequalities of Lemma A.2.

Proof. We have to show first that $\mu_n \leq n(n-1)$ ($\mu_n \geq 1$ is obvious). Consider the odd Δ case. Since μ_n is an increasing function of δ and the maximum value of δ is $d-1$, we see $\mu_n \leq \lfloor d(d-3+(d(d-3)-1)/((d-3)(d^2+d)))((d^2+d-1)/(2d-1)) \rfloor + 1$, but $d(d-3+(d(d-3)-1)/((d-3)(d^2+d)))((d^2+d-1)/(2d-1)) < d(d-3+1/d)d$, so the desired inequality follows from $d^2 - 3d + 1 < n$, $d < n$. The even Δ case is similar.

For the second inequality, we use a refined version of the proof of Corollary 4.1 of [HR1]. Consider the case in which Δ is odd, so $n = d^2 + 2\delta + 1$ and $r = d^2 + \delta$. We have to check that $(\mu_n - 1)(d^2 + \delta)/d + (d-3)/2 \geq (\mu_n - 1)\sqrt{n - 1/\mu_n}$, or equivalently,

$$(\mu_n - 1)\left(\sqrt{n - \frac{1}{\mu_n}} - \frac{d^2 + \delta}{d}\right) \leq \frac{d - 3}{2}.$$

Now $\sqrt{n - 1/\mu_n} \leq \sqrt{n} - 1/(2\mu_n\sqrt{n})$, so it will be enough to prove that

$$(\mu_n - 1)\left(\sqrt{n} - \frac{d^2 + \delta}{d}\right) \leq \frac{d - 3}{2} + \frac{\mu_n - 1}{2\mu_n\sqrt{n}}.$$

This is the same as $\mu_n - 1 \leq ((d-3)/2 + (1/(2\sqrt{n}))(1 - 1/\mu_n))(d^2/(d^2 - \delta^2))(\sqrt{n} + d + \delta/d)$. Taking into account that $d + (\delta+1)/d = (r+1)/d > \sqrt{n} > r/d = d + \delta/d$ and that $\mu_n \geq d(d-3)$ (because $d \geq 4$) it is enough to have $\mu_n - 1 \leq (d - 3 + (d(d-3)-1)/((d^2 + \delta + 1)(d-3)))d(d^2 + \delta)/(d^2 - \delta^2)$, which holds by hypothesis.

One handles the even case similarly, but now $n = d^2 + 2\delta$ and $r = d^2 + \delta - 1$, and $\delta > 0$ since n is not a square. $\square$

Proof of Theorem 2.1(e, f). By $(^{\circ\circ\circ})$, it is enough to prove $n\mu_n \geq \phi(n)$ for every nonsquare n, where μ_n is as in Lemma A.7, and where $\phi(n) = n^2$ if Δ is odd and $2d - 1 > \Delta \geq 4\sqrt[4]{n} + 1$, and where $\phi(n) = n(n\sqrt{n} - 5n + 5\sqrt{n} - 1)/2$ if $\Delta = 2d - 1$.

Suppose that Δ is odd with $\Delta \geq 4\sqrt[4]{n} + 1$, which implies that $\delta \geq 2\sqrt{d}$. We have to see that in this case $\mu_n \geq n$, so that we can take $\phi(n) = n^2$, as claimed. Consider the function

$$h(d, \delta) = d\left(d - 3 + \frac{d(d-3)-1}{(d-3)(d^2 + \delta + 1)}\right)\frac{d^2 + \delta}{d^2 - \delta^2} - n.$$

Substitute x^2 for d and $2x + t$ for δ, in which case $n = d^2 + 2\delta + 1 = x^4 + 4x + 2t + 1$. We want to show that $h \geq 0$ for $x \geq 2$ (i.e., for $d \geq 4$). By multiplying through to clear denominators, $h \geq 0$ becomes $p(x, t) \geq 0$, where $p(x, t) = x^2((x^2 - 3)^2(x^4 + 2x + t + 1) + x^2(x^2 - 3) - 1)(x^4 + 2x + t) - (x^4 + 4x + 2t + 1)(x^2 - 3)(x^4 + 2x + t + 1)(x^4 - (2x + t)^2)$. The partial $\partial p(x,t)/\partial t$ is $t^3(8x^2 - 24) + t^2(9x^6 - 27x^4 + 48x^3 + 9x^2 - 144x - 27) + t(2x^{10} - 6x^8 + 36x^7 + 2x^6 - 108x^5 + 84x^4 + 36x^3 - 268x^2 - 108x - 6) + (4x^{11} - x^{10} - 12x^9 + 33x^8 + 4x^7 - 91x^6 + 40x^5 + 36x^4 - 152x^3 - 100x^2 - 12x)$. For $x \geq \sqrt{6}$, it is not hard to check that the coefficients of the powers of t are all nonnegative. Thus $\partial p(x,t)/\partial t \geq 0$ for all $t \geq 0$ for each $x \geq \sqrt{6}$. Therefore, $p(x, t) \geq p(x, 0) \geq 0$. (In addition to $\delta \geq 2\sqrt{d}$ we also have $\delta \leq d - 1$, so in fact there are integers δ as above only if $d \geq 6$.)

Finally, suppose $n = d^2 + 2d - 1$, that is, $\Delta = 2\delta + 1$ with $\delta = d - 1$. Substituting the value of δ in the expression of μ_n we see that it is enough to verify

$$\frac{d^6 - 4d^5 - 2d^4 + 15d^3 + d^2 - 7d + 1}{(d+1)(d-3)(2d-1)} \geq \frac{1}{2}(n\sqrt{n} - 5n + 5\sqrt{n} - 1),$$

and the term on the left may be rewritten as

$$\frac{d^3 - 2d^2 - 4d}{2} + \frac{d^5 - 2d^4 - d^3 - 8d^2 - 2d + 2}{2(d+1)(d-3)(2d-1)}.$$

It is a straightforward computation that the second summand in the last expression is bounded below by 4 for $d \geq 4$. Now using the fact that $n = d^2 + 2d - 1$ and $\sqrt{n} \leq d + 1$ (and hence $n\sqrt{n} \leq d^3 + 3d^2 + d - 1$) the desired inequality follows. $\qquad \square$

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